Formation of Networks in a Context with Diversity

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Abstract

We present a model analyzing the endogenous network formation prior to an infinite-horizon network bargaining game. We assume agents of two types with either one of two alternatives: connections among players of the same type are cheaper than among players of different type or vice versa. In this way, players not only need to consider the trade-off between more outside options and the costs of maintaining those additional links, but also what type of players they connect to. We characterize pairwise stable network structures through necessary and sufficient conditions, highlighting the role played by the way in which heterogeneous nodes are placed in the different components for the pairwise stability of the networks. Finally, we perform a welfare analysis, comparing the efficient structures with those that are stable.

Keywords: Bargaining, Heterogeneity, Network formation.
JEL classification: C72, C78, D85

1 Introduction

Networks are used, among other things, to model situations in which pairs of connected agents may engage in exchange. It is easy to see that, for example, trading opportunities that depend on social relationships fit in this category. Traditionally, network models of bargaining assume homogeneous costs, not taking into account how diversity, language, customs or other considerations may affect these and can thus generate a propensity or difficulty (as in the case of languages or customs) to bargain with certain players. Examining only homogeneous costs does not show the complete picture, underestimating all the possible equilibrium structures. By considering heterogeneous costs instead, we take into account how things such as cultural and linguistic differences may affect the costs of bargaining with others while showing the various possible resulting equilibrium structures.

When networks representing bargaining opportunities are not exogenously given but endogenously determined, Gauer and Hellmann (2017) explain the trade-off faced by the agents of forming a new link: new connections affect the bargaining power because they increase the number of outside options but, on the flip side, they are costly to form and maintain. In our framework, we add one more layer: the cost of the link depends on the types of the connected players. Therefore, whether two players of different type will connect is determined by the entire network structure.

We study the endogenous network formation prior to a Manea (2011) infinite-horizon network bargaining game, assuming that there are agents of two types and that connections among players of the same type are cheaper than among players of different type. We call this the homophily case. Additionally, We also analyze the case in which connections between players of different types are more affordable than with players of the same type (e.g. after a subsidy from a government interested in diversifying its population). We refer to this as the heterophily case. The sufficient conditions characterizing the equilibrium network structures rely on the notion of pairwise stability introduced by Jackson and Wolinsky (1996).

We find that in the homophily case the stable components are isolated nodes, pairs, lines of length three, and odd cycles, just as in Gauer and Hellmann (2017). However, our framework allows us to highlight the role played by the way in which the heterogeneous nodes are placed in the different components for the pairwise stability of the network. For the heterophily case, instead, we find new structures such as stars and a slew of graphs from the bipartite family.

The rest of the paper is organized as follows: Section 2 describes the model. Section 3 provides the sufficient conditions in the homogeneous case for pairwise stable components and networks, emphasizing that only certain heterogeneous configurations can happen in equilibrium. Section 4 gives the sufficient conditions for the heterogeneous case for pairwise stability. In section 5 we present the welfare analysis for both cases. Section 6 concludes while the mathematical proofs are presented in appendix A.

1.1 Literature Review

Our paper contributes to the literature of bargaining in networks. The study of bargaining has a long history starting with the seminal papers by Nash (1950, 1953) and Rubinstein (1982). While the former follows an axiomatic approach, the latter attempts to look into the bargaining black-box, proposing a strategic model of alternating offers between two players whose results converge to those obtained by Nash when the players are infinitely patient.

Rubinstein and Wolinsky (1985) provide further insights into the process of decentralized bargaining in stationary markets, which would later allow for the study of bargaining in stationary networks. The authors assume that after a buyer and a seller reach an agreement they leave the market, and that this flow of departure is matched by an equal arrival flow of new agents of both types. But whereas in stationary networks the bargaining power is determined by the relative position of the nodes, in the absence of a network the driving force is the difference in the relative sizes of the two market sides.

The contributions to the literature of bargaining in exogenously-determined networks can be divided in two groups: those assuming stationary networks and those assuming non-stationary networks.

Stationary networks are those in which players reaching an agreement leave the market but are replaced by identical players in the subsequent period, and so the network structure remains the same.

Manea (2011) studies an infinite-horizon bargaining game in a stationary, undirected network. Although, contrary to our approach, the author considers the network structure as given, he shows that not all existing links will be used when players are patient enough; that is, there are pairs of players that, when selected to bargain, do not reach an agreement and instead prefer to wait to be matched with another trading partner. We make use of the algorithm developed in this paper to compute the equilibrium payoffs of the players when they are patient enough.

Gauer and Hellmann (2017) extend the model of Manea (2011) by considering the endogenous network formation in a stage prior to stationary bargaining. In their setup, players are ex ante homogeneous and sustaining the (undirected) links is costly. We generalize their framework by assuming that there can be two different types of players, and that the cost to sustain the links depends on the types of the players that connect.

Non-stationary networks are those in which players reaching an agreement leave the market but are not replaced. Their links in the network are also removed, so the network structure changes every time after an agreement.

Corominas-Bosch (2004) and Polanski (2007) study how centralized bargaining in networks work, with the latter generalizing the framework of the former by not limiting attention to bipartite graphs. Abreu and Manea (2012a,b) examine a model similar to Polanski (2007) but for decentralized bargaining, in which not all matchings necessarily lead to agreements and where multiple equilibria may exist.

2 The model

Assume a set of players $N = \{1, 2, ...n\}$ with $n \geq 3$ for a time period $t = 0, 1, 2, ...$ in which players interact. In the first period, $t = 0$, players form links in the network and from periods $t = 1, 2, ...$ they perform an infinite horizon bargaining game to split the unit surplus created from the links.

Denote a link between players $i, j \in N, i \neq j$ as $ij = ji = \{i, j\}$. Let g^N be the complete network, that is, the set of all subsets of N of size 2. Then the set of all networks that are undirected is $G = \left\{g \mid g \subseteq g^N\right\}$. We define the neighbors of player i as $N_i(g) = \{j \in N \mid ij \in N\}$ and let $\eta_i(g) = |N_i(g)|$ denote player i's *degree* (i.e. the cardinality of i's neighbors). Given a network g, a *path* between players i and j is a sequence of players $i_1, i_2, ..., i_M$ such that $i_m i_{m+1} \in g$ for all $m \in \{1, ...M-1\}$, with $i_1 = i$ and $i_M = j$. Let $C \subseteq N$ be a *component* of network g and we say that players $i, j \in C$ if there is a path between them in C, with $N_j(g) \cap C = \varnothing$ for all $j \neq C$. We say that a *subnetwork* $g' \subseteq g$ is *component-induced* if there is a component C of g such that $g'|g_C$ where the network defined as $g|_K = \{ij \in g \mid i, j \in K\}$ is the subnetwork limited to the players $K \subset N$. Given the networks $g, g' \subseteq g^N$, let $g + g' = g \cup g'$ represent the network obtained from adding the links $g' \setminus g$. Likewise, let $g - g' = g \setminus g'$ be the network obtained from severing links $g' \cap g$ from network g.

After the network formation stage, which happens at $t = 0$, bargaining takes place. This stage is modeled following Manea (2011): in periods $t = 1, 2, ...$, a uniform matching technology is assumed, which means that any link $ij \in g$ is randomly selected with probability p. Then, again with equal probability, one of the two players involved is selected as the proposer while the other is the responder. From the link that is selected, both players bargain about the surplus that is produced. The proposer makes an offer specifying how to divide the unit surplus from the link, while the responder either accepts it or rejects it. If the offer is rejected, both parties obtain zero as a payoff and continue in the game. If the offer is accepted, then both parties leave the game with the allocation that they agreed upon and they are replaced by new identical players at the exact same position, so that the network structure is not altered. All offers, responses and the network structure are considered to be common knowledge. It is important to note that players' payoffs are given by the discounted expected agreement share. In this game, players discount time by the discount factor $\delta \in (0, 1)$. Additionally, we say that a strategy profile is a sub-game perfect equilibrium of the game if it induces Nash equilibria in subgames following every history.

Manea (2011) finds that all sub-game perfect equilibria has an identical payoff. Furthermore, the payoff equilibrium for each player will depend only on that player's position in the network along with the discount factor δ. The unique solution to the equation system

$$v_i = \left(1 - \sum_{j \in N_i(g)} \frac{p}{2}\right) \delta v_i + \sum_{j \in N_i(g)} \frac{p}{2} \max\left(1 - \delta v_j, \delta v_i\right) \tag{1}$$

will be given by the *equilibrium payoff vector*, expressed by $v^{*\delta}(g) = \left(v_i^{*\delta}(g)\right)_{i \in N}$. Equation (1) must be satisfied by the equilibrium payoff because, by the stationarity assumption, strategies have to be such that a sub-game perfect equilibrium exists in which a given player i will always accept any offer for which he can obtain at least his continuation payoff, δv_i, and will always make the same proposals for identical responders. By offering δv_j to player j, player i will obtain $1 - \delta v_j$, which should not be less than his continuation payoff. Player i proposes to player j with probability $p/2$; that is, the probability of being chosen from all players in the network. From this we see that equilibrium payoffs have to satisfy (1) and are the unique solution to it; in other words, the solution is the unique fixed point. Thus, any equilibrium agreement between i and j is attainable if $\delta\left(v_i^{*\delta}(g) + v_j^{*\delta}(g)\right) \leq 1$ from which an *equilibrium agreement network* is defined as $g^{*\delta} = \left\{ij \in g \mid \delta_i^{*\delta}(g) + v_j^{*\delta}(g)\right\} \leq 1$.

We focus on the case when $\delta \to 1$, which means that players tend to be infinitely patient. In this way, equilibrium payoffs depend exclusively on the network structure. For large enough discount factors, Manea (2011) finds that the limit equilibrium agreement network does not change when δ does once g has been formed in the first stage; furthermore, the limit equilibrium payoff vector $v^*(g) = \lim_{\delta \to 1} v^{*\delta}(g)$ always exists.

We make use of the algorithm developed by Manea (2011) to compute the limit equilibrium payoff vector. Given a player set $M \subseteq N$ and a network g, the partner set in g is defined as $L^g(M) = \{j \in N | ij \in g, i \in M\}$. Finally, we say that a set $M \subseteq N$ is *g-independent* if no pair of players in M are connected in the network g. With these elements, the algorithm is defined as follows:

Definition 1 (Manea, 2011)**.** *For a given network g and player set N, the algorithm $\mathcal{A}(g)$ provides a sequence $(r_s, x_s, M_s, L_s, N_s, g_s)_{s=1,2,\ldots,\bar{s}}$ which is defined recursively as follows. Let $N_1 = N$ and $g_1 = g$. For $s \geq 1$, if $N_s = \varnothing$ then stop and set $\bar{s} = s$. Otherwise let*

$$r_s = \min_{M \subseteq N, M \in \mathcal{I}(g)} \frac{|L^{g_s}(M)|}{|M|}, \tag{2}$$

with $\mathcal{I}(g)$ denoting the set of nonempty g-independent sets.

If $r_s \geq 1$ then stop and set $\bar{s} = s$. Otherwise, set $x_s = r_s/(1 + r_s)$. Let M_s be the union of all minimizers in (2). Define $L_s = L^{g_s}(M_s)$. Let $N_{s+1} = N_s \setminus (M_s \cup L_s)$ and g_{s+1} be the subnetwork of g induced by players in N_{s+1}.

Let the result of $\mathcal{A}(g)$ be given by the sequence $(r_s, x_s, M_s, L_s, N_s, g_s)_{s=1,2,\ldots,\bar{s}}$. We say that the limit equilibrium payoffs are then given by:

$$v_i^* = x_s, \forall i \in M_s, \forall s < \bar{s}$$
$$v_j^* = 1 - x_s, \forall j \in L_s, \forall s < \bar{s}$$
$$v_k^* = \frac{1}{2}, \forall k \in N_{\bar{s}}$$

At each step s the algorithm $\mathcal{A}(g)$ searches for the *minimal shortage ratio* r_s from N_s (the remaining players N after each step) in the network $g_s = g|_{N_s}$, resulting from the largest g_s-independent set M_s to minimize $r_s = |L_s| / |M_s|$, with L_s representing the partner set of M_s. The smaller the composition of L_s relative to M_s is, the stronger the bargaining power that its members have to obtain bigger shares of the produced surplus. Relying on the algorithm, we can then find at each step the minimal shortage ratio and detect the players with the best and worst bargaining positions. The limit equilibrium payoffs for the players in M_s is given by $x_s = |L_s| / |L_s + M_s| = r_s/(1 + r_s)$ and for those in L_s is $1 - x_s = |M_s| / |L_s + M_s| = 1/(1 + r_s)$. It is easy to see that x_s is increasing in the shortage ratio. After each step, the matched players are removed from the game and the algorithm continues onward to the next step. This process continues until there are no more players left to match in the game or until $r_s \geq 1$. For this latter case, each of the remaining players receives the payoff $1/2$.

During the first stage, when the network is being formed, players foresee their payoffs from the bargaining game. We consider that there are two different types of players. Maintaining each link formed imposes a strictly positive cost to each of the players involved: this cost can be high, $\bar{c}$, or low, $\underline{c} < \bar{c}$. In particular, in the presence of homophily, the cost of creating a link between two nodes of the same type is $\underline{c}$ whereas it is $\bar{c}$ if the two nodes are of different types (and vice versa in the presence of heterophily).

At $t = 0$, each player tries to maximize their profit, expressed as

$$u_i^* (g) := v_i^* (g) - \overline{\eta}_i(g)\overline{c} - \underline{\eta}_i(g)\underline{c},$$

where $\overline{\eta}_i(g)$ represents the cardinality of the players of different type than i that are his neighbors, and $\underline{\eta}_i(g)$ denotes the cardinality of the players of the same type as i who are his neighbors.

Whether a link is created or not is determined by the resolution of the following trade-off: on the one hand, a link may benefit the involved players by altering their relative positions (and gross payoffs) in the network; on the other hand, a link is costly and this cost depends on the type of the connected players.

Following Gauer and Hellmann (2017), the profit profile $u^* = (u_i^*)_{i\in N}$ is said to be *component-decomposable* since $u_i^*(g) = u_i^*(g|_{C_{i(g)}})$ $\forall i \in N$ and g. $C_i \subseteq N$ represents the component of player i in network g, and we see that subnetworks induced by other components will not modify player i's profits as long as he does not belong to them.[1] As they did, we also rely on the equilibrium notion of Pairwise Stability and do not explicitly model the network formation in stage $t = 0$:

Definition 2 (Jackson and Wolinsky, 1996)**.** *A network g is pairwise stable if:*

1. *for all $ij \in g : u_i (g) \geq u_i (g - ij)$ and $u_j (g) \geq u_j (g - ij)$, and*
2. *for all $ij \notin g$: if $u_i (g + ij) > u_i (g)$, then $u_j (g + ij) < u_j (g)$*

The first part of the definition requires that no player wishes to delete a link that they are involved in; the second part requires that if some link is not in the network and one of the involved players would benefit from adding it, then it must be that the other player would suffer from the addition of the link. In other words, for a network to be pairwise stable, all players must desire to retain their existing links and no given pair of players wish to form a new one between themselves.

3 Homophily: Pairwise Stable Structures

We define *homophily* as the situation in which it is cheaper to create connections between nodes of the same type. This would reflect for instance that it is easier to do business among people who speak the same language or have similar backgrounds.

In this section we present sufficient conditions for pairwise stability. The two lemmas deal with the stability of individual components, whereas the three theorems characterize pairwise-stable networks with more than one component.

3.1 Lemmas

Lemma 1. *Sufficient conditions for components to be pairwise stable:*

[1] Notice that the algorithm $\mathcal{A}(g)$ assigns isolated players the payoff 0 because they simply cannot bargain.

(i) The homogeneous pair[2] *is pairwise stable if* $\underline{c} \leq \frac{1}{2}$.

The heterogeneous pair[3] *is pairwise stable if* $\overline{c} \leq \frac{1}{2}$.

(ii) The homogeneous line of length three[4] *is pairwise stable if* $\underline{c} = \frac{1}{6}$.

The heterogeneous line of length three such that two nodes of the same type are consecutive[5] *is pairwise stable if* $\overline{c} = \frac{1}{6}$.

(iii) The homogeneous odd cycle[6] *with at most* $\frac{1}{2\underline{c}}$ *players is pairwise stable if* $\underline{c} \leq \frac{1}{6}$.

The heterogeneous odd cycle[7] *with at most* $\frac{1}{2\overline{c}}$ *players is pairwise stable if* $\overline{c} \leq \frac{1}{6}$.

Proof.
In the Appendix. □

Lemma 2. *The following components are not pairwise stable:*

(i) Heterogeneous lines of length three such that two nodes of the same type are not consecutive.

(ii) Lines of length four or above.

(iii) Even cycles.

Proof.
In the Appendix. □

These results include those in Gauer and Hellmann (2017) as particular cases. There are two results worth a comment: the heterogeneous lines of length three, and the heterogeneous odd cycles.

Notice that, for heterogeneous lines of length three to be pairwise stable, it must be that the two nodes of the same type are consecutive. The intuition for this result is as follows: if the two nodes of the same type are not consecutive, the two existing links are expensive, which requires $\overline{c} \leq 1/6$. However, being of the same type, for the two peripheral nodes it is profitable to connect themselves if $\underline{c} \leq 1/6$, which is automatically implied by the previous condition; that is, in a line of length three, it is not possible to sustain two expensive links while preventing a cheap one.

On the other hand, when the two nodes of the same type are consecutive in the line of length three, there are a cheap link and an expensive link. Since the two peripheral nodes are different, the structure is stable if $\overline{c} = 1/6$, because that is the limiting value to simultaneously sustain the existing expensive link and prevent the

[2]The homogeneous pair is defined as two nodes of the same type connected.
[3]The heterogeneous pair is defined as two nodes of different types connected.
[4]The homogeneous line of length three is defined as a component of three players of the same type which can be transformed into a 3-player cycle by adding one link.
[5]The heterogeneous line of length three such that two nodes of the same type are consecutive is defined as a component of three players of different types which can be transformed into a 3-player cycle by adding one link between two players of different type.
[6]The homogeneous odd cycle is defined as an odd cycle in which all the nodes are of the same type.
[7]The heterogeneous odd cycle is defined as an odd cycle in which there are nodes of the two types.

formation of the cycle by adding an expensive link. Also, since the expensive link is maintained, so is the cheap existing one.

With respect to the heterogeneous odd cycles, the pairwise-stability condition is the same regardless of the number of nodes of each type and their positions in the cycle: since the cycle is odd, it is not profitable to add any link and so the only concern is to sustain the existing ones. In this structure, any pair of players involved in a link break would see themselves at the extremes of an odd line, and this symmetry explains why sustaining one expensive link in the cycle is not less demanding than sustaining more than one expensive link.

3.2 Theorems

Theorem 1 (Pairwise stability of the empty network)**.**
The empty network is pairwise stable if $\underline{c} \geq \frac{1}{2}$.

Proof.
In the Appendix. □

This result coincides with Gauer and Hellmann (2017): whereas in our setup the empty network may include nodes of different types, preventing the formation of a cheap link automatically prevents the formation of an expensive link, and so the relevant cost is $\underline{c}$ only.

Theorem 2 (Pairwise stable networks including only pairs and isolated nodes)**.**

(i) Networks consisting of the union of one isolated node and homogeneous pairs such that at least one is of the same type as the isolated, are pairwise stable if $\frac{1}{6} < \underline{c} \leq \frac{1}{2}$.

(ii) Networks consisting of the union of one isolated player and homogeneous pairs such that all are of the same type and different from the isolated, are pairwise stable if $\underline{c} \leq \frac{1}{2}$ *and* $\overline{c} > \max\left\{\frac{1}{6}, \underline{c}\right\}$.

(iii) Networks consisting of the union of homogeneous pairs, regardless of their types, and two isolated nodes such that each one of them is of a different type are pairwise stable if $\frac{1}{6} < \underline{c} \leq \frac{1}{2} \leq \overline{c}$. *Additionally, if* $\underline{c} = \frac{1}{2}$, *then there can exist two isolated nodes of the same type or three or more isolated nodes of any type.*

(iv) Networks consisting of the union of pairs such that at least one is heterogeneous and one isolated node are pairwise stable if $\frac{1}{6} < \underline{c} < \overline{c} \leq \frac{1}{2}$. *Additionally, if* $\frac{1}{6} < \underline{c} < \overline{c} = \frac{1}{2}$, *then there can exist two isolated nodes such that each one of them is of a different type.*

Proof.
In the Appendix. □

The first statement of Theorem 2 contains the result in Gauer and Hellmann (2017) as a particular case, while it also allows for the existence of homogeneous pairs different from the isolated player. However, in the second statement all pairs are homogeneous and different from the isolated node, which is reflected in the stability condition: to prevent the link between an isolated node and a pair of the same type, $1/6 < \underline{c}$ is sufficient, whereas to prevent the link between an isolated node and a pair of different type, the sufficient condition becomes $1/6 < \overline{c}$.

An important difference with Gauer and Hellmann (2017) is that there can be two isolated nodes rather than one in a network with only homogeneous pairs for different values of the costs within certain intervals. Of course, this happens because the condition to sustain the homogeneous pairs depends on $\underline{c}$ and the condition to prevent the link between the two isolated nodes that are of different types depends on $\overline{c}$. However, to allow for the existence of two or more isolated nodes of the same type, we need to fix the cheap cost at the particular limiting value $1/2$.

Finally, it is also possible to sustain heterogeneous pairs with one isolated node for different values of the costs within certain intervals. Nonetheless, in order to allow for two isolated nodes of different types to exist in this structure, the expensive cost needs to take the particular limiting value $1/2$. Notice that it is impossible to allow for three or more isolated nodes while sustaining heterogeneous pairs, because in that case it would be impossible to prevent the link between two isolated nodes of the same type.

Theorem 3 (Pairwise stable networks that include cycles).

(i) *Networks consisting of the union of homogeneous odd cycles, regardless of their types, with at most $\frac{1}{2\underline{c}}$ players, and two isolated nodes, one of each type, are pairwise stable if $\underline{c} \leq \frac{1}{6}$ and $\overline{c} \geq \frac{1}{2}$.*

(ii) *Networks consisting of the union of homogeneous cycles, regardless of their types, with at most $\frac{1}{2\underline{c}}$ players, and either homogeneous pairs, regardless of their types, or at most one isolated player are pairwise stable if $\underline{c} \leq \frac{1}{6}$. Additionally, if $\underline{c} = \frac{1}{6}$ and given that there is no isolated player, then there can also be homogeneous lines of length three, regardless of their types.*

(iii) *Networks consisting of the union of homogeneous cycles, regardless of their types, with at most $\frac{1}{2\underline{c}}$ players, and pairs such that at least one of them is heterogeneous are pairwise stable if $\underline{c} \leq \frac{1}{6}$ and $\overline{c} \leq \frac{1}{2}$. Additionally, if $\underline{c} = \frac{1}{6}$, then there can also be homogeneous lines of length three, regardless of their types.*

(iv) *Networks consisting of the union of homogeneous odd cycles, regardless of their types, with at most $\frac{1}{2\underline{c}}$ players, one isolated player and homogeneous pairs such that all are of the same type and different from the isolated node are pairwise stable if $\underline{c} \leq \frac{1}{6} < \overline{c}$. Additionally, if $\underline{c} = \frac{1}{6}$, then there can also be homogeneous lines of length three such that all are of the same type and different from the isolated node.*

(v) *Networks consisting of the union of odd cycles such that at least one is heterogeneous, with at most* $\frac{1}{2\underline{c}}$ *players in the homogeneous cycle(s) and at most* $\frac{1}{2\overline{c}}$ *players in the heterogeneous cycle(s), and either pairs or at most one isolated player are pairwise stable if* $\overline{c} \leq \frac{1}{6}$.

(vi) *Networks consisting of the union of odd cycles, with at most* $\frac{1}{2\underline{c}}$ *players in the homogeneous cycle(s) and 3 players in the heterogeneous cycle(s), and one heterogeneous line of length three are pairwise stable if* $\overline{c} = \frac{1}{6}$. *Additionally, if* $\frac{1}{15} < \underline{c}$, *then there can also be pairs.*

Proof.
In the Appendix. □

Notice that the first statement of Theorem 3 cannot be obtained in Gauer and Hellmann (2017), because if a homogeneous odd cycle is sustainable, then the two isolated nodes of the same type will form a pair. However, if the isolated nodes are of different types, the sufficient condition to sustain the homogeneous odd cycle depends on $\underline{c}$ whereas the sufficient condition to prevent the formation of a heterogeneous pair depends on $\overline{c}$.

The second statement of Theorem 3 includes the results in Gauer and Hellmann (2017) as a particular case, while it is not restricted to structures in which all pairs and cycles are of the same type (just being homogeneous is enough).

The lines of length three deserve further attention, as their existence in the pairwise stable network structures limits the characteristics of the other components. In particular, homogeneous lines can co-exist with isolated nodes of different type, pairs, other homogeneous lines and homogeneous cycles with three players, but not with heterogeneous cycles. On the other hand, heterogeneous lines cannot co-exist with other lines or isolated players, but they can with pairs and cycles. Moreover, they limit the number of players in the heterogeneous cycles to three, but they do not limit the number of players in the homogeneous cycles.

Finally, it is important to remark that there are multiple equilibria for certain parametric conditions. For example, if $\underline{c} \leq 1/6 \wedge 1/6 < \overline{c} \leq 1/2$, a network structure with the characteristics defined in Theorem 2 *(ii)* is pairwise stable, and so is a network structure with the characteristics defined in Theorem 3 *(iii)*. Our results simply state that the two configurations are pairwise stable, but nothing is said about which configuration would be finally reached, or with which probability.

4 Heterophily: Pairwise Stable Structures

We define *heterophily* as the situation in which it is cheaper to create connections between nodes of different types. This would reflect for instance that there are positive spillovers when people who are experts on different fields cooperate in the same research project, or simply the case of buyer-seller networks.

In this section we present sufficient conditions for pairwise stability. The lemma below deals with the stability of individual components.

4.1 Lemmas

Lemma 3. *Sufficient conditions for components to be pairwise stable:*

(i) *The homogeneous pair is pairwise stable if* $\bar{c} \leq \frac{1}{2}$.
The heterogeneous pair is pairwise stable if $\underline{c} \leq \frac{1}{2}$.

(ii) *The homogeneous line of length three is pairwise stable if* $\bar{c} = \frac{1}{6}$.
The heterogeneous line of length three such that two nodes of the same type are not consecutive is pairwise stable if $\underline{c} \leq \frac{1}{6} \leq \bar{c}$.

(iii) *A star with* n *leaves, all of the same type and different from the root,*[8] *is pairwise stable if* $\underline{c} \leq \frac{1}{n(n+1)}$ *and* $\bar{c} \geq \frac{n-1}{2(n+1)}$.

(iv) *Odd lines such that all nodes in odd positions are of one type and all nodes in even positions are of the other type are pairwise stable if* $\underline{c} \leq (m - \widetilde{m})/2m\widetilde{m} < 1/2m \leq \bar{c}$.[9]

(v) *The odd cycles with at most* $\frac{1}{2\bar{c}}$ *players, regardless of whether they are homogeneous or heterogeneous, are pairwise stable if* $\bar{c} \leq \frac{1}{6}$.

Proof.
In the Appendix. □

There are other bipartite graphs that are stable. Their complete characterization is still Work in Progress.

5 Conclusions

We have presented a model to analyze the endogenous network formation prior to a Manea (2011) infinite-horizon network bargaining game, assuming that there are agents of two types and that either the connections among players of the same type are cheaper than among players of different types or vice versa. This consideration adds one more layer to the basic trade-off pointed out by Gauer and Hellmann (2017): additional links provide more outside options but they are costly, and this cost depends on the characteristics of the agents involved in the formation of the link.

Key results regarding the pairwise stability of components refer to heterogeneous lines of length three and heterogeneous odd cycles. While for heterogeneous lines of length three to be pairwise stable it must be that the two nodes of the same type are consecutive, for heterogeneous odd cycles to be pairwise stable it does not matter either how many nodes of each type there are or their positions in the cycle.

[8] The heterogeneous line of length three such that two nodes of the same type are not consecutive is a particular case of this star.

[9] The heterogeneous line of length three such that two nodes of the same type are not consecutive is a particular case.

Key results regarding the pairwise stability of network structures refer to the components that can co-exist. Whereas pairs and cycles, both homogeneous and heterogeneous, co-exist in many configurations, the existence of lines of length three imposes further constraints. In particular, homogeneous lines can co-exist with isolated nodes of different type, pairs, other homogeneous lines and homogeneous cycles with three players, but not with heterogeneous cycles. On the other hand, heterogeneous lines cannot co-exist with other lines or isolated players, but they can with pairs and cycles. Moreover, they limit the number of players in the heterogeneous cycles to three, but they do not limit the number of players in the homogeneous cycles.

Although in this paper we have considered just two different types of agents, it would be interesting to extend the model to r types of agents and c_1, c_2, ..., c_r different linking costs. This consideration is left for future research.

A Appendix

Proof. Lemma 3.1:

(i) Pairs: according to the definition of Pairwise Stability, we only need to find the condition for the existing link to be kept.

Homogeneous: the cost for each player to sustain the link is $\underline{c}$. When broken, they become isolated and get 0. Then, $1/2 - \underline{c} \geq 0 \Leftrightarrow \underline{c} \leq 1/2$. Therefore, the homogeneous pair is pairwise stable if $\underline{c} \leq \frac{1}{2}$.

Heterogeneous: the cost for each player to sustain the link is $\overline{c}$. When broken, they become isolated ad get 0. Then, $1/2 - \overline{c} \geq 0 \Leftrightarrow \overline{c} \leq 1/2$. Therefore, the heterogeneous pair is pairwise stable if $\overline{c} \leq \frac{1}{2}$.

(ii) Lines of length three: according to the definition of Pairwise Stability, we need to check that the two links connecting each peripheral player with the central node are kept, and that the peripheral players do not want to create a link among themselves.

Homogeneous: the cost for each player to sustain a link is $\underline{c}$. When broken, the central node becomes part of a pair, receiving the gross payoff 1/2, and the peripheral node becomes isolated, getting 0. Then, for the link to be sustained:

· central node: $\frac{2}{3} - \underline{c} \geq \frac{1}{2} \Leftrightarrow \underline{c} \leq \frac{1}{6}$.

· peripheral node: $\frac{1}{3} - \underline{c} \geq 0 \Leftrightarrow \underline{c} \leq \frac{1}{3}$.

Thus, the link is sustained if $\underline{c} \leq 1/6$.

The cost for each peripheral player to form a new link is $\underline{c}$. Then, as the payoffs are the same for both peripheral nodes,

· create if $\frac{1}{2} - \underline{c} > \frac{1}{3} \Leftrightarrow \underline{c} < \frac{1}{6}$.

Thus, the link is not created if $\underline{c} \geq 1/6$.

Therefore, the homogeneous line of length three is pairwise stable if $\underline{c} = \frac{1}{6}$.

Heterogeneous: the cost for players of the same type to sustain a link is $\underline{c}$, while for players of different types it is $\overline{c}$. When broken, the central node becomes part of a pair, receiving the gross payoff 1/2, and the peripheral node becomes isolated, getting 0. Then,

- central node with peripheral node of the same type:
  - · central node: $\frac{2}{3} - \underline{c} \geq \frac{1}{2} \Leftrightarrow \underline{c} \leq \frac{1}{6}$.
  - · peripheral node: $\frac{1}{3} - \underline{c} \geq 0 \Leftrightarrow \underline{c} \leq \frac{1}{3}$.

  The link is sustained if $\underline{c} \leq 1/6$.
- central node with peripheral node of different type:
  - · central node: $\frac{2}{3} - \overline{c} \geq \frac{1}{2} \Leftrightarrow \overline{c} \leq \frac{1}{6}$.
  - · peripheral node: $\frac{1}{3} - \overline{c} \geq 0 \Leftrightarrow \overline{c} \leq \frac{1}{3}$.

  The link is sustained if $\overline{c} \leq 1/6$.

The cost for each peripheral player to form a new link is $\overline{c}$, in which case an odd cycle is created and each player receives the gross payoff 1/2. Then, as the payoffs are the same for both peripheral nodes,

- · create if $\frac{1}{2} - \overline{c} > \frac{1}{3} - \Leftrightarrow \overline{c} < \frac{1}{6}$.

Thus, the link is not created if $\overline{c} \geq 1/6$.

Therefore, the heterogeneous line of length three such that two nodes of the same type are consecutive is pairwise stable if $\overline{c} = \frac{1}{6}$.

(iii) Odd cycles: according to the definition of Pairwise Stability, we need to find the conditions for each node to keep the links with its two neighbors, and for no additional link to be created.

Homogeneous: the cost for each player to sustain a link is $\underline{c}$. If the link is broken, each node ends up at one of the extremes of an odd line of length $\underline{m}$, receiving the gross payoff $(\underline{m} - 1)/2\underline{m}$. Then,

$$\frac{1}{2} - \underline{c} \geq \frac{\underline{m} - 1}{2\underline{m}} \Leftrightarrow \underline{c} \leq \frac{1}{2\underline{m}}$$

As $\underline{m} \geq 3$, $\underline{c} \leq \frac{1}{6}$.

Notice that, after creating a link between two nodes of the cycle that were unconnected, the gross payoff for each player remains $\frac{1}{2}$, as the cycle is odd. Then, these players were strictly better off without the additional link for any $\underline{c} > 0$.

Therefore, the homogeneous odd cycle with at most $\frac{1}{2\underline{c}}$ players is pairwise stable if $\underline{c} \leq \frac{1}{6}$.

Heterogeneous: the cost for players of the same type to sustain a link is $\underline{c}$, while for players of different types it is $\overline{c}$. If the link is broken, each node ends up at one of the extremes of an odd line of length $\overline{m}$, receiving the gross payoff $(\overline{m}-1)/2\overline{m}$. Sustaining links between players of different types automatically guarantees that links between players of the same type are sustained as well. Also, notice that in heterogeneous odd cycles there is always at least one link that costs $\overline{c}$ for the players involved. Then,

$$\frac{1}{2} - \overline{c} \geq \frac{\overline{m}-1}{2\overline{m}} \Leftrightarrow \overline{c} \leq \frac{1}{2\overline{m}}.$$

As $\overline{m} \geq 3$, $\overline{c} \leq \frac{1}{6}$.

Again, after creating a link between two nodes of the cycle that were unconnected, the gross payoff for each player remains $\frac{1}{2}$, as the cycle is odd. Then, these players were strictly better off without the additional link for any $\underline{c} > 0$.

Therefore, the heterogeneous cycle with at most $\frac{1}{2\overline{c}}$ players is pairwise stable if $\overline{c} \leq \frac{1}{6}$.

Notice that the pairwise stability condition does not depend on the number of nodes of each type or on their positions within the heterogeneous cycle.

□

Proof. Lemma 3.2:

(i) Heterogeneous lines of length three such that two nodes of the same type are not consecutive.

In this structure, every existing link costs $\overline{c}$ to each player. If any link is broken, the central player becomes part of a pair, receiving the gross payoff 1/2, and the peripheral player becomes isolated, getting the payoff 0. Then,

- central node: $\frac{2}{3} - \overline{c} \geq \frac{1}{2} \Leftrightarrow \overline{c} \leq \frac{1}{6}$.
- peripheral node: $\frac{1}{3} - \overline{c} \geq 0 \Leftrightarrow \overline{c} \leq \frac{1}{3}$.

The links are sustained if $\overline{c} \leq 1/6$.

The cost for each peripheral player to form a new link is $\underline{c}$, in which case an odd cycle is created and each player receives the gross payoff 1/2. Then, as the payoffs are the same for both peripheral nodes,

- create if $\frac{1}{2} - \underline{c} > \frac{1}{3} \Leftrightarrow \underline{c} < \frac{1}{6}$.

The link is not created if $\underline{c} \geq 1/6$, but this condition contradicts $\overline{c} \leq 1/6$, and so the structure is not pairwise stable.

(ii) Lines of length four or above.

We first focus on the even lines. In these structures, each node receives the gross payoff $\frac{1}{2}$. Consider the node connected to a peripheral player. It has to sustain two links in the even line, whereas if it kept the connection with the peripheral player only, it would still receive the gross payoff $\frac{1}{2}$ while sustaining just one link, which makes it strictly better off for any positive value of the cost.

We now focus on the odd lines with five or more nodes, m. In these structures, the set M is composed of the odd nodes, so $|M| = (m+1)/2$. Therefore, $|L| = (m-1)/2$ and $r = |L|/|M| = (m-1)/(m+1) < 1$. Accordingly, the gross payoff of the odd nodes is $x_{odd} = r/(1+r) = (m-1)/2m$ and the gross payoff of the even nodes is $x_{even} = 1/(1+r) = (m+1)/2m$.

Consider the link between a peripheral player and the subsequent node. If it breaks, the peripheral node becomes isolated, so getting the payoff 0, and the interior node becomes the extreme of an even line, so receiving the gross payoff $\frac{1}{2}$. The link with a peripheral player of the same type is sustained if $(m+1)/2m - \underline{c} \geq 1/2 \Leftrightarrow \underline{c} \leq 1/2m$ (and if $\overline{c} \leq 1/2m$ when the peripheral player is of different type).

Consider now the link between two interior nodes. When it breaks, the node occupying the even position in the odd line becomes the extreme node of an even line, receiving the gross payoff 1/2, whereas the node occupying the odd position in the odd line turns into the extreme node of a new odd line of length $\widetilde{m} < m$, getting the gross payoff $(\widetilde{m}-1)/2\widetilde{m}$. When the two interior nodes are of the same type, the even node keeps the link if $(m+1)/2m - \underline{c} \geq 1/2 \Leftrightarrow \underline{c} \leq 1/2m$, whereas the odd node keeps the link if $(m-1)/2m - \underline{c} \geq (\widetilde{m}-1)/2\widetilde{m} \Leftrightarrow \underline{c} \leq (m-\widetilde{m})/2m\widetilde{m}$. Thus, the link is sustained if $\underline{c} \leq \min\left\{\frac{1}{2m}, \frac{m-\widetilde{m}}{2m\widetilde{m}}\right\}$. Analogously, when the two interior nodes are of different types, the link is sustained if $\overline{c} \leq \min\left\{\frac{1}{2m}, \frac{m-\widetilde{m}}{2m\widetilde{m}}\right\}$.

Take an odd line such that the two peripheral nodes are of the same type. If the line connecting the two peripheral nodes were created, an odd cycle would result, implying that each node receives the gross payoff 1/2. Thus, the link will not be created if $\underline{c} \geq 1/2m$. If the line is homogeneous, keeping all the links requires $\underline{c} \leq (m-\widetilde{m})/2m\widetilde{m} < 1/2m$ (the last strict inequality coming from the fact that $m \geq 5$), which contradicts $\underline{c} \geq 1/2m$. If the line is heterogeneous, keeping all the links requires $\underline{c} < \overline{c} \leq \min\left\{\frac{1}{2m}, \frac{m-\widetilde{m}}{2m\widetilde{m}}\right\}$, which contradicts $\underline{c} \geq 1/2m$.

Take an odd line such that the two peripheral nodes are of different types. Notice that it is always possible to connect one of the peripheral nodes with an interior odd node of the same type. If such a link were created, the resulting structure would be an odd cycle connected to an even line and each player would get the gross payoff 1/2. Thus, such a link will not be created if $\underline{c} \geq$

$1/2m$. As this line is by definition heterogeneous, keeping all links requires $\underline{c} < \overline{c} \leq \min\left\{\frac{1}{2m}, \frac{m-\widetilde{m}}{2m\widetilde{m}}\right\}$, which contradicts $\underline{c} \geq 1/2m$.

(iii) Even cycles.

In these structures, each player receives the gross payoff $1/2$. However, if a link is broken, the resulting structure is an even line and, again, each player gets the gross payoff $1/2$. Therefore, the players that broke the link are strictly better off in the line, as they sustain a single link rather than two for any strictly positive value of the costs.

□

Proof. Theorem 3.1:
Suppose that there are three or more nodes. In that case, there will always be at least two nodes of the same type. Then, $\underline{c} \geq 1/2$ is sufficient to prevent the creation of any link: preventing the connection between two nodes of the same type also prevents the connection between two nodes of different types, as the former link is cheaper than the latter. □

Proof. Theorem 3.2:
Notice the following:

(a) Two pairs never create a link to connect themselves. The reason is that the gross payoff of the nodes creating the link remains $1/2$, so the new structure makes them strictly worse off for any strictly positive value of the costs, as they have to sustain two links rather than one.

(b) To prevent the formation of a link between an isolated node and the extreme of a pair that is of the same type, $1/6 < \underline{c}$ is required (as for the node belonging to the pair is not profitable to become the central node of a line of length three if $1/2 > 2/3 - \underline{c} \Leftrightarrow \underline{c} > 1/6$). Analogously, to prevent the formation of a link between an isolated and the extreme node of a pair that is of different type, $1/6 < \overline{c}$ is required.

Consider a network with the characteristics specified in (i). The condition to sustain the homogeneous pairs is $\underline{c} \leq 1/2$, as it was stated in Lemma 1. To prevent the formation of a link between the isolated node and a pair that is of the same type, $1/6 < \underline{c}$ is required. Since $\overline{c} > \underline{c}$, the previous condition also guarantees that a link between the isolated node and a pair that is of different type will not be formed. The intersection of all the conditions is $\frac{1}{6} < \underline{c} \leq \frac{1}{2} \wedge \overline{c} > \underline{c}$, which makes the network pairwise stable.

Consider a network with the characteristics specified in (ii). Again, the condition to sustain the homogeneous pairs is $\underline{c} \leq 1/2$. However, since there are no pairs of the same type as the isolated node, only $1/6 < \overline{c}$ is required to prevent the formation of links. As by assumption $\overline{c} > \underline{c}$, the intersection of all the conditions is $\underline{c} \leq \frac{1}{2} \wedge \overline{c} > \max\left\{\frac{1}{6}, \underline{c}\right\}$, which makes the network pairwise stable.

Consider a network with the characteristics specified in the first part of statement (*iii*). Again, the condition to sustain the homogeneous pairs is $\underline{c} \leq 1/2$. As in (*i*), $1/6 < \underline{c}$ is sufficient to prevent the formation of a link between an isolated node and a pair of the same type, and it also automatically prevents the creation of a link between an isolated node and a pair of different type. Finally, to prevent the formation of a link between the two isolated nodes that are of different types, $1/2 \leq \overline{c}$ is required. The intersection of all conditions is $\frac{1}{6} < \underline{c} \leq \frac{1}{2} \leq \overline{c} \wedge \overline{c} > \underline{c}$, which makes the network pairwise stable.
Consider the particular case $\frac{1}{6} < \underline{c} = \frac{1}{2} < \overline{c}$. $\underline{c} = 1/2$ is the intersection between $\underline{c} \leq 1/2$, sufficient to sustain homogeneous pairs, and $\underline{c} \geq 1/2$, sufficient to prevent the link between two isolated nodes of the same type. Plus, as $\overline{c} > \underline{c}$, $\overline{c} > \underline{c} \geq 1/2$ also prevents the link between two isolated nodes of different types (notice that, whenever there are three or more isolated nodes, there are at least two of them of the same type).

Finally, consider a network with characteristics specified in the first part of statement (*iv*). In this case, the condition to sustain the heterogeneous pair(s) is $\overline{c} \leq 1/2$, which also sustains the homogeneous pairs, if there is any. Notice that, as there is at least one heterogeneous pair, there is always a node of the same type as the isolated that belongs to a pair. The link between these two nodes is prevented if $1/6 < \underline{c}$ (which is also sufficient to prevent the link between the isolated node and a node of different type that belongs to a pair). The intersection of all conditions is $\frac{1}{6} < \underline{c} < \overline{c} \leq \frac{1}{2}$, which makes the network pairwise stable.
Consider the particular case $\frac{1}{6} < \underline{c} < \overline{c} = \frac{1}{2}$. $\overline{c} = 1/2$ is the intersection between $\overline{c} \leq 1/2$, sufficient to sustain heterogeneous pairs, and $\overline{c} \geq 1/2$, sufficient to prevent the link between two isolated nodes of different types, which allows to introduce one more isolated node different from the previous one. Notice that introducing one more isolated node of the same type is not pairwise stable: as $\underline{c} \leq 1/2$, a link between them will be created. □

Proof. Theorem 3.3:
Recall from the proof of Theorem 2 that two pairs do not create a link among themselves; that $1/6 < \underline{c}$ is sufficient to prevent the formation of a link between an isolated node and the extreme of a pair that is of the same type; and that $1/6 < \overline{c}$ is sufficient to prevent the formation of a link between an isolated node and the extreme of a pair that is of different type.

Also, notice that:

(a) An odd cycle never creates a link with another structure, including an isolated node, because the player in the cycle creating such a link keeps receiving the gross payoff 1/2 while paying to sustain three links rather than two.

(b) A link between a homogeneous line of length three and an isolated node of different type can be prevented if $1/6 = \underline{c} < \overline{c}$. If the line is either heterogeneous or homogeneous but of the same type as the isolated node, the creation of a link with the isolated player cannot be prevented. The reason is that the link between an isolated node and the extreme of a length-3 line that is of the same type is not created if $\underline{c} > 1/6$. In the former case, the homogeneous line is sustained if $\underline{c} = 1/6$, which contradicts $\underline{c} > 1/6$. In the latter case, the heterogeneous line is sustained if $\bar{c} = 1/6$, which again contradicts $\underline{c} > 1/6$ because $\bar{c} > \underline{c}$ by assumption.

(c) A pair and a length-3 line such that they have extreme nodes of the same type do not create a link if $\underline{c} > 1/15$. Analogously, a pair and a length-3 line such that they do not have extreme nodes of the same type do not create a link if $\bar{c} > 1/15$. Also, notice that the central node of a line of length three does not create a link with a pair for any strictly positive level of the costs, as its gross payoff remains invariant after the creation of such a link.

(d) Two homogeneous length-3 lines of the same type do not create a link if $\underline{c} \geq 1/6$; analogously, two homogeneous length-3 lines of different types do not create a link if $\bar{c} \geq 1/6$. On the contrary, a link between a heterogeneous length-3 line and another length-3 line cannot be prevented. The reason is that a heterogeneous line always has one extreme that can connect with an extreme of the same type of another line (regardless of whether this line is homogeneous or heterogeneous). This link is not created if $\underline{c} \geq 1/6$, but the heterogeneous line is sustained if $\bar{c} = 1/6$, which contradicts the previous condition because $\bar{c} > \underline{c}$ by assumption.

Consider a network with the characteristics specified in (i). Homogeneous odd cycles with at most $1/2\underline{c}$ players are stable if $\underline{c} \leq 1/6$. When adding two isolated nodes, each of a different type, the whole network is pairwise stable by just preventing the link between them, as the cycles do not connect either with an isolated node or among themselves. Then, the network is pairwise stable if $\underline{c} \leq \frac{1}{6} \wedge \bar{c} \geq \frac{1}{2}$. Notice that either pairs or lines of length three cannot fit in this network, as a link with the isolated node of the same type as the extreme node is not created if $\underline{c} > 1/6$, which makes the cycles unstable.

Consider a network with the characteristics specified in the fist part of statement (ii). Homogeneous odd cycles with at most $1/2\underline{c}$ players are stable if $\underline{c} \leq 1/6$. This condition automatically allows to add to the network either one isolated node or homogeneous pair(s): cycles do not create links with any other component, and $\underline{c} \leq 1/6$ is stricter that $\underline{c} \leq 1/2$, which is the sufficient condition to sustain homogeneous pairs. Then, a network with these components is pairwise stable if $\underline{c} \leq \frac{1}{6} \wedge \bar{c} > \underline{c}$. Note that if there was at least one homogeneous pair of the same type as the isolated node, they would connect as $\underline{c} \leq 1/6$. See case (iv) for the pairwise stable conditions when all pairs are homogeneous and different from the isolated node.
Consider the particular case $\underline{c} = \frac{1}{6} < \bar{c}$. If the network is only composed of homogeneous odd cycles and homogeneous pairs, then there can also be homogeneous lines of length three: cycles and pairs keep being stable, lines are sustained and they do not create links either between themselves ($\underline{c} = 1/6 \geq 1/6$) or with pairs

$(1/15 < \underline{c} = 1/6 < \overline{c})$. However, if the network is composed of homogeneous odd cycles and one isolated node, homogeneous length-3 lines in general do not fit: as long as there is one line of the same type as the isolated player, a link between them cannot be prevented (as $\underline{c} = 1/6$ contradicts $\underline{c} > 1/6$). See case (iv) for the pairwise stability conditions when all lines are homogeneous and different from the isolated node.

Consider a network with the characteristics specified in the first part of statement (iii). Homogeneous odd cycles with at most $1/2\underline{c}$ players are stable if $\underline{c} \leq 1/6$, and heterogeneous pairs are stable if $\overline{c} \leq 1/2$ (which makes homogeneous pairs also sustainable, as $\overline{c} > \underline{c}$). Then, the network is pairwise stable if $\underline{c} \leq \frac{1}{6} \wedge \underline{c} < \overline{c} \leq \frac{1}{2}$. Consider the particular case $\frac{1}{6} = \underline{c} < \overline{c} \leq \frac{1}{2}$. In this case, homogeneous lines of length three also fit in this network: cycles and pairs keep being stable, lines are sustained and they do not create links either between themselves ($\underline{c} = 1/6 \geq 1/6$) or with pairs ($1/15 < \underline{c} = 1/6$).

Consider a network with the characteristics specified in the first part of statement (iv). Homogeneous odd cycles with at most $1/2\underline{c}$ players are stable if $\underline{c} \leq 1/6$, and so are homogeneous pairs as $\underline{c} \leq 1/6$ is stricter than $\underline{c} \leq 1/2$. If $\overline{c} > 1/6$, given that all homogeneous pairs are of the same type, an isolated node of different type fits, since $\overline{c} > 1/6$ prevents any link between these pairs and the isolated player. Then, the network is pairwise stable if $\underline{c} \leq \frac{1}{6} < \overline{c}$. Notice that if there was at least one heterogeneous pair or one homogeneous pair of the same type as the isolated node, the network would not be pairwise stable because the condition to prevent the link between one of these pairs and the isolated player is $\underline{c} > 1/6$, which contradicts the condition to sustain the cycles.
Consider the particular case $\underline{c} = \frac{1}{6} < \overline{c}$. If this is the case, homogeneous length-3 lines such that all are of the same type and different from the isolated node also fit: cycles and pairs keep being stable, lines are sustained, neither lines nor pairs create a link with the isolated player as $\overline{c} > 1/6$, lines do not create links between themselves because $\underline{c} = 1/6 \geq 1/6$, and lines and pairs do not connect because $1/15 < \underline{c} = 1/6$.

Consider a network with the characteristics specified in (v). Heterogeneous odd cycles with at most $1/2\overline{c}$ players are stable if $\overline{c} \leq 1/6$, and so are homogeneous odd cycles with at most $1/2\underline{c}$ players since $\overline{c} > \underline{c}$. This condition automatically allows to add to the network either one isolated node or pair(s), which can be homogeneous and/or heterogeneous: cycles do not create links with any other component and $\overline{c} \leq 1/6$ is stricter than both $\overline{c} \leq 1/2$ and $\underline{c} \leq 1/2$, which are the sufficient conditions to sustain heterogeneous and homogeneous pairs, respectively. Then, a network with these components is pairwise stable if $\underline{c} < \overline{c} \leq \frac{1}{6}$. Notice that, for these parametric conditions, a link between a pair and an isolated node cannot be prevented as $\underline{c} < \overline{c} \leq 1/6$ contradicts both $\overline{c} > 1/6$ and $\underline{c} > 1/6$, so these components cannot co-exist in this network.

Finally, consider a network with the characteristics specified in the first part of statement (vi). Heterogeneous odd cycles with 3 players are stable if $\overline{c} = 1/6$, which also sustains the heterogeneous line of length three. Since $\overline{c} > \underline{c}$ by assumption, homogeneous cycles with at most $1/2\underline{c}$ are also allowed, so the network is pairwise stable if $\underline{c} < \overline{c} = \frac{1}{6}$. Notice that more length-3 lines do not fit: as $\underline{c} < 1/6$, their

extremes would always connect.
Consider the particular case $\frac{1}{15} < \underline{c} < \overline{c} = \frac{1}{6}$. Then, pairs homogeneous and/or heterogeneous also fit: $\overline{c} = 1/6$ is stricter than both $\overline{c} \leq 1/2$ and $\underline{c} \leq 1/2$, so allowing for pairs, and $1/15 < \underline{c}$ prevents the creation of links between the heterogeneous line and a pair. However, since $1/15 < \underline{c}$, the number of players of the homogeneous odd cycle(s) can never be larger than seven. □

Proof. Lemma 4.1:

(i) Pairs: according to the definition of Pairwise Stability, we only need to find the condition for the existing link to be kept.

Homogeneous: the cost for each player to sustain the link is $\overline{c}$. When broken, they become isolated and get 0. Then, $1/2 - \overline{c} \geq 0 \Leftrightarrow \overline{c} \leq 1/2$. Therefore, the homogeneous pair is pairwise stable if $\overline{c} \leq \frac{1}{2}$.

Heterogeneous: the cost for each player to sustain the link is $\underline{c}$. When broken, they become isolated ad get 0. Then, $1/2 - \underline{c} \geq 0 \Leftrightarrow \underline{c} \leq 1/2$. Therefore, the heterogeneous pair is pairwise stable if $\underline{c} \leq \frac{1}{2}$.

(ii) Lines of length three: according to the definition of Pairwise Stability, we need to check that the two links connecting each peripheral player with the central node are kept, and that the peripheral players do not want to create a link among themselves.

Homogeneous: the cost for each player to sustain a link is $\overline{c}$. When broken, the central node becomes part of a pair, receiving the gross payoff 1/2, and the peripheral node becomes isolated, getting 0. Then, for the link to be sustained:

· central node: $\frac{2}{3} - \overline{c} \geq \frac{1}{2} \Leftrightarrow \overline{c} \leq \frac{1}{6}$.
· peripheral node: $\frac{1}{3} - \overline{c} \geq 0 \Leftrightarrow \overline{c} \leq \frac{1}{3}$.

Thus, the link is sustained if $\overline{c} \leq 1/6$.

The cost for each peripheral player to form a new link is $\overline{c}$. Then, as the payoffs are the same for both peripheral nodes,

· create if $\frac{1}{2} - \overline{c} > \frac{1}{3} \Leftrightarrow \overline{c} < \frac{1}{6}$.

Thus, the link is not created if $\overline{c} \geq 1/6$.

Therefore, the homogeneous line of length three is pairwise stable if $\overline{c} = \frac{1}{6}$.

Heterogeneous: the cost for players of the different types to sustain a link is $\underline{c}$. When broken, the central node becomes part of a pair, receiving the gross payoff 1/2, and the peripheral node becomes isolated, getting 0. Then, for the link to be sustained:

· central node: $\frac{2}{3} - \underline{c} \geq \frac{1}{2} \Leftrightarrow \underline{c} \leq \frac{1}{6}$.
· peripheral node: $\frac{1}{3} - \underline{c} \geq 0 \Leftrightarrow \underline{c} \leq \frac{1}{3}$.

Thus, the link is sustained if $\underline{c} \leq 1/6$.

The cost for each peripheral player to form a new link is $\overline{c}$, in which case an odd cycle is created and each player receives the gross payoff 1/2. Then, as the payoffs are the same for both peripheral nodes,

- create if $\frac{1}{2} - \overline{c} > \frac{1}{3} \Leftrightarrow \overline{c} < \frac{1}{6}$

Thus, the link is not created if $\overline{c} \geq 1/6$.

Therefore, the heterogeneous line of length three such that two nodes of the same type are not consecutive is pairwise stable if $\underline{c} \leq \frac{1}{6} \leq \overline{c}$.

(iii) Stars with $n \geq 2$ leaves, all of the same type and different from the root: we first notice that all the links of this structure are cheap and that all leaves get the same payoff. Therefore, Pairwise Stability requires that cheap links are sustained and that an expensive link connecting the leaves is prevented.

When an existing link is broken, the root remains in its position but with one partner less, receiving the gross payoff $(n-1)/n$, and the leaf becomes isolated, getting 0. Then, for the link to be sustained:

- root: $\frac{n}{n+1} - \underline{c} \geq \frac{n-1}{n} \Leftrightarrow \underline{c} \leq \frac{1}{n(n+1)}$.
- leaf: $\frac{1}{n+1} - \underline{c} \geq 0 \Leftrightarrow \underline{c} \leq \frac{1}{n+1}$.

Thus, the link is sustained if $\underline{c} \leq \frac{1}{n(n+1)}$.

When two leaves connect, an odd cycle results and each leaf gets the gross payoff 1/2. As the payoffs are the same for the connecting leaves,

- create if $\frac{1}{2} - \overline{c} > \frac{1}{n+1} \Leftrightarrow \overline{c} < \frac{n-1}{2(n+1)}$.

Thus, the link is not created if $\overline{c} \geq \frac{n-1}{2(n+1)}$. Therefore, the star with $n \geq 2$ leaves of the same type and different from the root is pairwise stable if $\underline{c} \leq \frac{1}{n(n+1)} \leq \frac{n-1}{2(n+1)} \leq \overline{c}$.

(iv) Odd lines such that all nodes in odd positions are of one type and all nodes in even positions are of the other type: given the alternating types of the nodes, all the existing links are cheap. Furthermore, the creation of odd cycles implies establishing an expensive link, which has to be deterred for the structure to be pairwise stable.[10]

Let us focus on the odd lines with five or more nodes, $m \geq 5$, as the line of three nodes has been discussed previously. In these structures, the set M is composed of the odd nodes, so $|M| = (m+1)/2$. Therefore, $|L| = (m-1)/2$ and $r = |L|/|M| = (m-1)/(m+1) < 1$. Accordingly, the gross payoff of the odd nodes is $x_{odd} = r/(1+r) = (m-1)/2m$ and the gross payoff of the even nodes is $x_{even} = 1/(1+r) = (m+1)/2m$.

[10] Even cycles are never created.

Consider the link between a peripheral player and the subsequent node. If it breaks, the peripheral node becomes isolated, so getting the payoff 0, and the interior node becomes the extreme of an even line, so receiving the gross payoff $\frac{1}{2}$. The link is sustained if $(m+1)/2m - \underline{c} \geq 1/2 \Leftrightarrow \underline{c} \leq 1/2m$.

Consider now the link between two interior nodes. When it breaks, the node occupying the even position in the odd line becomes the extreme node of an even line, receiving the gross payoff 1/2, whereas the node occupying the odd position in the odd line turns into the extreme node of a new odd line of length $\widetilde{m} < m$, getting the gross payoff $(\widetilde{m}-1)/2\widetilde{m}$. The even node keeps the link if $(m+1)/2m - \underline{c} \geq 1/2 \Leftrightarrow \underline{c} \leq 1/2m$, whereas the odd node keeps the link if $(m-1)/2m - \underline{c} \geq (\widetilde{m}-1)/2\widetilde{m} \Leftrightarrow \underline{c} \leq (m-\widetilde{m})/2m\widetilde{m}$. Thus, the link is sustained if $\underline{c} \leq \min\left\{\frac{1}{2m}, \frac{m-\widetilde{m}}{2m\widetilde{m}}\right\}$.

Notice that, as $m \geq 5$, to keep all the links requires $\underline{c} \leq (m-\widetilde{m})/2m\widetilde{m} < 1/2m$.

Regarding the creation of a new link such that an odd cycle results, notice that only the nodes in odd positions could be benefited by it. For the nodes in even positions, the gross payoff would not change after the creation of the link, so they would be strictly worse off.

When two nodes occupying odd positions create a link, each one of them gets the gross payoff 1/2, but such a link is expensive. Then,

- create if $\frac{1}{2} - \overline{c} > \frac{m-1}{2m} \Leftrightarrow \overline{c} < \frac{1}{2m}$

Thus, the link is not created if $\overline{c} \geq 1/2m$, and the entire structure is pairwise stable if $\underline{c} \leq (m-\widetilde{m})/2m\widetilde{m} < 1/2m \leq \overline{c}$.

(v) Odd cycles: according to the definition of Pairwise Stability, we need to find the conditions for each node to keep the links with its two neighbors, and for no additional link to be created.

Since the cycles are odd, regardless of whether they are homogeneous or heterogeneous, there will be two consecutive nodes of the same type; that is, there exists at least one expensive link that, if sustained, implies the sustainability of the cheap links as well.

If such a link is broken, each node ends up at one of the extremes of an odd line of length $\overline{m}$, receiving the gross payoff $(\overline{m}-1)/2\overline{m}$. Then,

$$\frac{1}{2} - \overline{c} \geq \frac{\overline{m}-1}{2\overline{m}} \Leftrightarrow \overline{c} \leq \frac{1}{2\overline{m}}.$$

As $\overline{m} \geq 3$, $\overline{c} \leq \frac{1}{6}$.

After creating a link between two nodes of the cycle that were unconnected, the gross payoff for each player remains $\frac{1}{2}$, as the cycle is odd. Then, these players were strictly better off without the additional link for any $\underline{c} > 0$.

Therefore, an odd cycle with at most $\frac{1}{2\overline{c}}$ players is pairwise stable if $\overline{c} \leq \frac{1}{6}$.

□

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